

# THE STEIN STRÖMBERG COVERING THEOREM IN METRIC SPACES

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**ABSTRACT.** In [NaTa] Naor and Tao extended to the metric setting the  $O(d \log d)$  bounds given by Stein and Strömberg for Lebesgue measure in  $\mathbb{R}^d$ , deriving these bounds first from a localization result, and second, from a random Vitali lemma. Here we show that the Stein-Strömberg original argument can also be adapted to the metric setting, giving a third proof. We also weaken the hypotheses, and additionally, we sharpen the estimates for Lebesgue measure.

## 1. INTRODUCTION

In [StSt], Stein and Strömberg proved that for Lebesgue measure in  $\mathbb{R}^d$ , and with balls defined by an arbitrary norm, the centered maximal function has weak type (1,1) bounds of order  $O(d \log d)$ , which is much better than the exponential bounds obtained via the Vitali covering lemma. Naor and Tao extended the Stein-Strömberg result to the metric setting in [NaTa]. There, a localization result is proven (using the notion of microdoubling, which basically entails a very regular growth of balls) from which the Stein-Strömberg bounds are obtained (using the notion of strong microdoubling, which combines microdoubling with local comparability). Also, a second argument is given, via a random Vitali Theorem that has its origin in [Li].

Here we note that the Stein-Strömberg original proof, which is shorter and conceptually simpler, can also be used in the metric setting, yielding a slightly more general result. We will divide the Stein-Strömberg argument into two parts, one with radii separated by large gaps, and the second, with radii inside an interval, bounded away from 0 and  $\infty$ . This will allow us to obtain more precise information about which hypotheses are needed in each case. We shall see that under the same hypotheses used by Naor and Tao, the Stein-Strömberg covering theorem for sparse radii (cf. Theorem 4.1 below) suffices to obtain the  $d \log d$  bounds in the metric setting. But Theorem 4.1 itself is presented in a more general version. In particular, one does not need to assume that the metric space is geometrically doubling.

We also show that the Stein-Strömberg method, applied to balls with no restriction in the radii, yields the  $O(d \log d)$  bounds in the metric context, for doubling measures where the growth of balls can be more irregular than is allowed by the microdoubling condition. Finally, we lower the known weak type (1,1) bounds in the case of Lebesgue measure: For  $d$  lacunary sets of radii, from  $(e^2 + 1)(e + 1)$  to  $(e^{1/d} + 1)(1 + 2e^{1/d})$  (to 6 in the specific case of  $\ell_\infty$  balls),

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and for unrestricted radii, from  $e^2(e^2 + 1)(1 + o(1))d \log d$  to  $(2 + 3\varepsilon)d \log d$ , where  $\varepsilon > 0$  and  $d = d(\varepsilon)$  is sufficiently large.

## 2. NOTATION AND BACKGROUND MATERIAL

Some of the definitions here come from [A2]; we refer the interested reader to that paper for motivation and additional explanations.

We will use  $B(x, r) := \{y \in X : d(x, y) < r\}$  to denote open balls,  $\overline{B(x, r)}$  to denote their topological closure, and  $B^{cl}(x, r) := \{y \in X : d(x, y) \leq r\}$  to refer to closed balls. Recall that in a general metric space, a ball  $B$ , considered as a set, can have many centers and many radii. When we write  $B(x, r)$  we mean to single out  $x$  and  $r$ , speaking respectively of the center and the radius of  $B(x, r)$ .

**Definition 2.1.** A Borel measure is  $\tau$ -smooth if for every collection  $\{U_\alpha : \alpha \in \Lambda\}$  of open sets,  $\mu(\cup_\alpha U_\alpha) = \sup \mu(\cup_{i=1}^n U_{\alpha_i})$ , where the supremum is taken over all finite subcollections of  $\{U_\alpha : \alpha \in \Lambda\}$ . We say that  $(X, d, \mu)$  is a *metric measure space* if  $\mu$  is a Borel measure on the metric space  $(X, d)$ , such that for all balls  $B(x, r)$ ,  $\mu(B(x, r)) < \infty$ , and furthermore,  $\mu$  is  $\tau$ -smooth.

The assumption of  $\tau$ -smoothness does not represent any real restriction, since it is consistent with standard set theory (Zermelo-Fraenkel with Choice) that in every metric space, every Borel measure which assigns finite measure to balls is  $\tau$ -smooth (cf. [Fre, Theorem (a), pg. 59]).

**Definition 2.2.** Let  $(X, d, \mu)$  be a metric measure space and let  $g$  be a locally integrable function on  $X$ . For each  $x \in X$ , the centered Hardy-Littlewood maximal operator  $M_\mu$  is given by

$$(1) \quad M_\mu g(x) := \sup_{\{r: 0 < \mu(B(x, r))\}} \frac{1}{\mu(B(x, r))} \int_{B(x, r)} |g| d\mu.$$

Maximal operators can be defined using closed balls instead of open balls, and this does not change their values, as can be seen by an approximation argument. When the measure is understood, we will omit the subscript  $\mu$  from  $M_\mu$ .

A sublinear operator  $T$  satisfies a weak type  $(1, 1)$  inequality if there exists a constant  $c > 0$  such that

$$(2) \quad \mu(\{Tg > s\}) \leq \frac{c\|g\|_{L^1(\mu)}}{s},$$

where  $c = c(T, \mu)$  depends neither on  $g \in L^1(\mu)$  nor on  $s > 0$ . The lowest constant  $c$  that satisfies the preceding inequality is denoted by  $\|T\|_{L^1 \rightarrow L^{1, \infty}}$ .

**Definition 2.3.** A Borel measure  $\mu$  on  $(X, d)$  is *doubling* if there exists a  $C > 0$  such that for all  $r > 0$  and all  $x \in X$ ,  $\mu(B(x, 2r)) \leq C\mu(B(x, r)) < \infty$ .

**Definition 2.4.** A metric space is *D-geometrically doubling* if there exists a positive integer  $D$  such that every ball of radius  $r$  can be covered with no more than  $D$  balls of radius  $r/2$ .

If a metric space supports a doubling measure, then it is geometrically doubling. Regarding weak type inequalities for the maximal operator, in order to estimate  $\mu\{Mf > s\}$ , one considers balls  $B(x, r)$  over which  $|f|$  has average larger than  $s$ . Now, while in the uncentered case any such ball is contained in the corresponding level set, this is not so for the centered maximal function. Thus, using the balls  $B(x, r)$  to cover  $\{Mf > s\}$  can be very inefficient.

A key ingredient in the Stein-Strömberg proof is to cover  $\{Mf > s\}$  by the much smaller balls  $B(x, tr)$ ,  $0 < t \ll 1$ , something that leads to the “microdoubling” notion of Naor and Tao. We slightly modify their notation, using  $1/n$ -microdoubling to denote what these authors call  $n$ -microdoubling.

**Definition 2.5.** ([NaTa, p. 735] Let  $0 < t < 1$  and let  $K \geq 1$ . Then  $\mu$  is said to be  $t$ -microdoubling with constant  $K$  if for all  $x \in X$  and all  $r > 0$ , we have

$$\mu B(x, (1+t)r) \leq K\mu B(x, r).$$

The next property is mentioned in [NaTa], and more extensively studied in [A2].

**Definition 2.6.** A measure  $\mu$  satisfies a *local comparability condition* if there exists a constant  $C \in [1, \infty)$  such that for all pairs of points  $x, y \in X$ , and all  $r > 0$ , whenever  $d(x, y) < r$ , we have

$$\mu(B(x, r)) \leq C\mu(B(y, r)).$$

We denote the smallest such  $C$  by  $C(\mu)$  or  $C_\mu$ .

**Remark 2.7.** If  $\mu$  is doubling with constant  $K_1$  then it is microdoubling and satisfies a local comparability condition with the same constant  $K_1$ , while if it is  $t$ -microdoubling with constant  $K_2$  and  $2 \leq (1+t)^M$ , then  $\mu$  is doubling and satisfies a local comparability condition with constant  $K_2^M$ . Thus, the difference between doubling and microdoubling lies in the size of the constants, it is quantitative, not qualitative: The microdoubling condition adds something new only when  $K_2 < K_1$ , in which case it entails a greater regularity in the growth of the measure of balls, as the radii increase. Likewise, bounds of the form  $\mu B(x, Tr) \leq K\mu B(x, r)$  for  $T > 2$ , allow a greater irregularity in the growth of balls than standard doubling ( $T = 2$ ) or than microdoubling.

We mention that while local comparability is implied by doubling, it is a uniformity condition, not a growth condition. Thus, it is compatible with the failure of doubling, and even for doubling measures, it is compatible with any rate of growth for the volume of balls. Consider, for instance, the case of  $d$ -dimensional Lebesgue measure  $\lambda^d$ : A doubling constant is  $2^d$ , a  $1/d$ -microdoubling constant is  $e$ , and the smallest local comparability constant is  $C(\lambda^d) = 1$ .

The next definition combines the requirement that the microdoubling and the local comparability constants be “small” simultaneously.

**Definition 2.8.** ([NaTa, p. 737] Let  $0 < t < 1$  and let  $K \geq 1$ . Then  $\mu$  is said to be strong  $t$ -microdoubling with constant  $K$  if for all  $x \in X$ , all  $r > 0$ , and all  $y \in B(x, r)$ ,

$$\mu B(y, (1+t)r) \leq K\mu B(x, r).$$

Thus, if  $\mu$  is strong  $t$ -microdoubling with constant  $K$ , then  $C(\mu) \leq K$ . Also, local comparability is the same as strong 0-microdoubling. To get a better understanding of how bounds depend on the different constants, it is useful to keep separate  $C(\mu)$  and  $K$ .

**Definition 2.9.** Given a set  $S$  we define its  $s$ -blossom as the enlarged set

$$(3) \quad Bl(S, s) := \cup_{x \in S} B(x, s),$$

and its *uncentered*  $s$ -blossom as the set

$$(4) \quad Blu(S, s) := \cup_{x \in S} \cup \{B(y, s) : x \in B(y, s)\}.$$

When  $S = B(x, r)$ , we simplify the notation and write  $Bl(x, r, s)$ , instead of  $Bl(B(x, r), s)$ , and likewise for uncentered blossoms. We say that  $\mu$  *blossoms boundedly* if there exists a  $K \geq 1$  such that for all  $r > 0$  and all  $x \in X$ ,  $\mu(Blu(x, r, r)) \leq K\mu(B(x, r)) < \infty$ .

Blossoms can be defined using closed instead of open balls, in an entirely analogous way. To help understand the relationship between blossoms and balls, we include the following definitions and results.

**Definition 2.10.** A metric space has the *approximate midpoint property* if for every  $\varepsilon > 0$  and every pair of points  $x, y$ , there exists a point  $z$  such that  $d(x, z), d(z, y) < \varepsilon + d(x, y)/2$ .

**Definition 2.11.** A metric space  $X$  is *quasiconvex* if there exists a constant  $C \geq 1$  such that for every pair of points  $x, y$ , there exists a curve with  $x$  and  $y$  as endpoints, such that its length is bounded above by  $Cd(x, y)$ . If for every  $\varepsilon > 0$  we can take  $C = 1 + \varepsilon$ , then we say that  $X$  is a *length space*.

It is well known that for a complete metric space, having the approximate midpoint property is equivalent to being a length space.

**Example 2.12.** The  $s$ -blossom of an  $r$ -ball may fail to contain a strictly larger ball, even in quasiconvex spaces.

For instance, let  $X \subset \mathbb{R}^2$  be the set  $\{0\} \times [0, 1] \cup [0, 1] \times \{0\}$  with metric defined by restriction of the  $\ell_\infty$  norm; then we can take  $C = 2$ . Now  $B((1, 0), 1) = (0, 1] \times \{0\}$ , while for every  $r > 1$ ,  $B((1, 0), r) = X$ , which is not contained in  $Blu((1, 0), 1, 1/6)$ . Furthermore, neither  $Blu((1, 0), 1, 1/6)$  nor  $Bl((1, 0), 1, 1/6)$  are balls, i.e., given any  $x \in X$  and any  $r > 0$ , we have that  $B(x, r) \neq Blu((1, 0), 1, 1/6)$  and  $B(x, r) \neq Bl((1, 0), 1, 1/6)$ .

On the other hand, if a metric space  $X$  has the approximate midpoint property, then blossoms and balls coincide (as we show next) so in this case considering blossoms gives nothing new.

**Theorem 2.13.** Let  $(X, d)$  be a metric space. The following are equivalent:

- a)  $X$  has the approximate midpoint property.
- b) For all  $x \in X$ , and all  $r, s > 0$ ,  $Bl(x, r, s) = B(x, r + s)$ .
- c) For all  $x \in X$ , and all  $r > 0$ ,  $Bl(x, r, r) = B(x, 2r)$ .

*Proof.* Suppose first that  $X$  has the approximate midpoint property. Since  $Bl(x, r, s) \subset B(x, r + s)$ , to prove b) it is enough to show that if  $y \in B(x, r + s)$ , then  $y \in Bl(x, r, s)$ , or equivalently, that there is a  $z \in X$  such that  $d(x, z) < r$  and  $d(z, y) < s$ . If either  $d(x, y) < s$  or  $d(x, y) < r$  we can take  $z = x$  and there is nothing to prove, so assume otherwise. Let  $(\hat{X}, \hat{d})$  be the completion of  $(X, d)$ ; then  $\hat{X}$  is a length space, since it has the approximate midpoint property. Let  $\Gamma : [0, 1] \rightarrow \hat{X}$  be a curve with  $\Gamma(0) = x$ ,  $\Gamma(1) = y$ , and length  $\ell(\Gamma) < r + s$ . Then  $\Gamma([0, 1]) \subset B(x, r) \cup B(y, s)$ , for if there is a  $w \in [0, 1]$  with  $\Gamma(w) \notin B(x, r) \cup B(y, s)$ , then  $\ell(\Gamma) \geq r + s$ . Now let  $c \in [0, 1]$  be the time of first exit of  $\Gamma(t)$  from  $B(x, r)$ , that is, for all  $t < c$ ,  $\Gamma(t) \in B(x, r)$  and  $\Gamma(c) \notin B(x, r)$ . Then  $\Gamma(c) \in B(y, s)$ , so by continuity of  $\Gamma$ , there is a  $\delta \in [0, c]$  such that  $\Gamma(\delta) \in B(y, s)$ . Thus, the open set  $B(x, r) \cap B(y, s) \neq \emptyset$  in  $\hat{X}$ . But  $X$  is dense in  $\hat{X}$ , so there exists a  $z \in X$  such that  $d(x, z) < r$  and  $d(z, y) < s$ , as we wanted to show.

Part c) is a special case of part b). From part c) we obtain a) as follows. Let  $x, y \in X$ , and let  $r > 0$  be such that  $d(x, y) < 2r$ . By hypothesis,  $y \in Bl(x, r, r) = B(x, 2r)$ , so there is a  $z \in X$  such that  $d(x, z) < r$  and  $d(z, y) < r$ . Thus,  $X$  has the approximate midpoint property.  $\square$

**Example 2.14.** Let  $X$  be the unit sphere (unit circumference) in the plane, with the chordal metric, that is, with the restriction to  $X$  of the euclidean metric in the plane. While this space does not have the approximate midpoint property, blossoms are nevertheless geodesic balls. However, the equality  $Bl(x, r, s) = B(x, r + s)$  no longer holds. For instance,  $Bl((1, 0), 1, 1) \neq Bl((1, 0), \sqrt{2}, \sqrt{2}) = B((1, 0), 2) = X \setminus \{(-1, 0)\}$ .

### 3. MICROBLOSSOMING AND RELATED CONDITIONS

**Definition 3.1.** Let  $0 < t < 1$  and let  $K \geq 1$ . Then  $\mu$  is said to  $t$ -microblossom boundedly with constant  $K$ , if for all  $x \in X$  and all  $r > 0$ , we have

$$(5) \quad \mu(Blu(x, r, tr)) \leq K\mu B(x, r).$$

We shall say  $\mu$  is a measure that  $(t, K)$ -microblossoms, instead of using the longer expression.

**Example 3.2.** Microblossoming (even together with doubling) is more general than microdoubling, in a quantitative sense. Consider  $(\mathbb{Z}^d, \ell_\infty, \mu)$ , where  $\mu$  is the counting measure. Then  $\mu$  is doubling, and “microdoubling in the large”, since for large radii ( $r > d$ ),  $\mu$  can be regarded as a discrete approximation to Lebesgue measure. However,  $\mu B(0, 1) = 1$ , and for every  $t > 0$ ,  $\mu B(0, 1 + t) \geq 3^d$ , no matter how small  $t$  is. Thus, the measure  $\mu$  is not  $(t, K)$ -microdoubling, for any  $K < 3^d$ ,  $0 < t < 1$ . However,  $\mu$  is  $1/d$ -microblossoming, since for  $r > d$ ,  $\mu$  behaves as a microdoubling measure, and for  $r \leq d$ ,  $Blu(x, r, r/d) = B(x, r)$ .

A less natural but stronger example is furnished by the measure  $\mu$  given by [A2, Theorem 5.9]. Since  $\mu$  satisfies a local comparability condition, and is defined in a geometrically doubling space, it blossoms boundedly, so it microblossoms boundedly (at least with the blossoming constant). But  $\mu$  is not doubling, and hence it is not microdoubling.

**Example 3.3.** While  $(t, K_1)$ -microdoubling entails  $(2, K_2)$ -doubling for some  $K_2 \geq K_1$ , the analogous statement is not true for microblossoming. The following example shows that  $(1/2, 1)$ -microblossoming does not entail local comparability. Let  $X = \{0, 1, 3\}$  with the inherited metric from  $\mathbb{R}$ , and let  $\mu = \delta_3$ . Then  $B(0, 3) \cap B(3, 3) = \{1\}$ , but  $\mu B(0, 3) = 0$  while  $\mu B(3, 3) = 1$ , so local comparability fails. Since bounded blossoming implies local comparability, all we have to do is to check that  $\mu$  is  $(1/2, 1)$ -microblossoming. For  $t \leq 3$ ,  $B(0, t) \subset Blu(0, t, t/2) \subset \{0, 1\}$ , so  $\mu B(0, t) = \mu Blu(0, t, t/2) = 0$ , and for  $t > 3$ ,  $B(0, t) = Blu(0, t, t/2) = X$ . Likewise, for  $t \leq 2$ ,  $B(1, t) = Blu(1, t, t/2) \subset \{0, 1\}$ , so  $\mu B(1, t) = \mu Blu(1, t, t/2) = 0$ , and for  $t > 2$ ,  $B(1, t) = Blu(1, t, t/2) = X$ .

**Definition 3.4.** Given a metric measure space  $(X, d, \mu)$ , and denoting the support of  $\mu$  by  $\text{supp}(\mu)$ , the *relative increment function* of  $\mu$ ,  $ri_\mu : \text{supp}(\mu) \times (0, \infty) \times [1, \infty)$ , is defined as

$$(6) \quad ri_\mu(x, r, t) := \frac{\mu B(x, tr)}{\mu B(x, r)},$$

and the *maximal relative increment function*, as

$$(7) \quad mri_\mu(r, t) := \sup_{x \in \text{supp}(\mu)} \frac{\mu B(x, tr)}{\mu B(x, r)}.$$

When  $\mu$  is understood we will simply write  $ri$  and  $mri$ .

This notation allows one to unify different conditions that have been considered regarding the boundedness of maximal operators. For instance, on  $\text{supp}(\mu)$  the doubling condition simply means that there is a constant  $C \geq 1$  such that for all  $r > 0$ ,  $mri_\mu(r, 2) \leq C$ , and the  $d^{-1}$ -microdoubling condition, that for all  $r > 0$ ,  $mri_\mu(r, d^{-1}) \leq C$ . Note that by  $\tau$ -smoothness, the complement of the support of  $\mu$  has  $\mu$ -measure zero, so the relative increment function is defined for almost every  $x$ .

**Example 3.5.** The interest of considering values of  $t > 2$  in the preceding definition comes from the fact that, under the additional assumption of microblossoming, it will allow a much more irregular growth of balls than microdoubling or plain doubling, without a comparable worsening of the estimates for the weak type  $(1, 1)$  bounds.

To fix ideas, consider the right hand side  $C(\mu) K_1 K \left(2 + \frac{\log K_2}{\log K}\right)$  of formula (10) below. This bound is related to the centered maximal operator when the supremum is restricted to radii  $R$  between  $r$  and  $Tr$ ,  $T > 1$ . The constant  $K_2$  depends on  $T$ , as it must satisfy  $mri_\mu(r, T) \leq K_2$ . For Lebesgue measure on  $\mathbb{R}^d$  with the  $\ell_\infty$ -norm,  $C(\lambda^d) = 1$ . If we set  $T = 2$ , then we can take  $K_2 = 2^d$ , while  $K_2 = d^d$  for  $T = d$ , a choice which yields bounds of order  $O(d \log d)$ . A  $1/d$ -microdoubling constant is  $K_1 = e$  ( $\mathbb{R}^d$  has the approximate midpoint property, and in fact it is a geodesic space, so microdoubling is the same as microblossoming in this case) and  $K := \max\{K_1, e\} = e$ .

Returning to Example 3.2, by a rescaling argument it is clear that the situation for  $(\mathbb{Z}^d, \ell_\infty, \mu)$  cannot be much worse than for  $(\mathbb{R}^d, \ell_\infty, \lambda^d)$ , and in fact it is easy to see that the same argument of Stein and Strömberg (which will be presented in greater generality

below) yields the  $O(d \log d)$  bounds. Now suppose we modify the measure so that at one single point it is much smaller. Clearly, this will have little impact in the weak type (1,1) bounds, since for  $d \gg 1$ ,  $x \in \mathbb{Z}^d$ , and  $r > 1$ , the measure of  $B(x, r)$  will be changed by little or not at all, while for  $r \leq 1$ , balls with distinct centers do not intersect. For definiteness, set  $\nu = \mu$  on  $\mathbb{Z}^d \setminus \{0\}$ , and  $\nu\{0\} = d^{-d}$ . Then the doubling constant, and the  $(t, K)$ -microdoubling constant, for any  $t > 0$ , is at least  $d^d(3^d - 1) \leq K = K_2$ , much larger than the corresponding constants for  $\mu$ . However, the local comparability constant is still very close to 1, since intersecting balls of the same radius must contain at least  $3^d$  points each, and a  $1/d$ -microblossoming constant can be taken to be very close to  $e$ . Setting  $T = d$ , we get  $K_2 \leq d^d(2d + 1)^d$ , so  $\log K_2$  in this case is comparable to the constant obtained when  $T = 2$ .

**Remark 3.6.** One might define  $(T, K)$ -macroblossoming, with  $T > 1$ , by analogy with Definition 3.1. However, since  $B(x, Tr) \subset Blu(x, r, Tr)$ , assuming directly that  $mri_\mu(r, T) \leq K$  is not stronger than  $(T, K)$ -macroblossoming,

#### 4. THE STEIN-STRÖMBERG COVERING THEOREM

Next, we present the Stein-Strömberg argument using the terminology of blossoms. Note that the next theorem does not require  $X$  to be geometrically doubling.

Given an ordered sequence of sets  $A_1, A_2, \dots$ , we denote by  $D_1, D_2, \dots$  its sequence of disjointifications, that is  $D_1 = A_1$ , and  $D_{n+1} = A_{n+1} \setminus \cup_1^n A_i$ . We shall avoid reorderings and relabelings of collections of balls, as this may lead to confusion regarding the meaning of  $D_j$ . The unfortunate downside of this choice is an inflation of subindices.

**Theorem 4.1. Stein-Strömberg covering theorem for sparse radii.** *Let  $(X, d, \mu)$  be a metric measure space, where  $\mu$  satisfies a  $C(\mu)$  local comparability condition, and let  $R := \{r_n : n \in \mathbb{Z}\}$  be a  $T$ -lacunary sequence of radii, i.e.,  $r_n > 0$  and  $r_{n+1}/r_n \geq T > 1$ . Suppose there exists a  $t > 0$  such that  $Tt \geq 1$  and  $\mu$   $t$ -microblossoms boundedly with constant  $K$ . Let  $\{B(x_i, s_i) : s_i \in R, 1 \leq i \leq M\}$  be a finite collection of balls with positive measure, ordered by non-increasing radii. Set  $U := \cup_{i=1}^M B(x_i, ts_i)$ . Then there exists a subcollection  $\{B(x_{i_1}, s_{i_1}), \dots, B(x_{i_N}, s_{i_N})\}$ , such that, denoting by  $D_{i_j}$  the disjointifications of the reduced balls  $B(x_{i_j}, ts_{i_j})$ ,*

$$(8) \quad \mu U \leq (K + 1) \mu \cup_{j=1}^N B(x_{i_j}, ts_{i_j}),$$

and

$$(9) \quad \sum_{j=1}^N \frac{\mu D_{i_j}}{\mu B(x_{i_j}, s_{i_j})} \mathbf{1}_{B(x_{i_j}, s_{i_j})} \leq C(\mu) K + 1.$$

*Proof.* We use the Stein-Strömberg selection algorithm. Let  $B(x_{i_1}, s_{i_1}) = B(x_1, s_1)$  and suppose that the balls  $B(x_{i_1}, s_{i_1}), \dots, B(x_{i_k}, s_{i_k})$  have already been selected. If

$$\sum_{j=1}^k \frac{\mu D_{i_j}}{\mu B(x_{i_j}, s_{i_j})} \mathbf{1}_{Bl(x_{i_j}, s_{i_j}, ts_{i_j})}(x_{i_{k+1}}) \leq 1,$$

accept  $B(x_{i_{k+1}}, s_{i_{k+1}}) := B(x_{i_k+1}, s_{i_k+1})$  as the next ball in the subcollection. Otherwise, reject it. Repeat till we run out of balls. Let  $\mathcal{C}$  be the collection of all rejected balls. Then  $\mu$  a.e.,

$$\mathbf{1}_{\cup \mathcal{C}} < \sum_{j=1}^N \frac{\mu D_{i_j}}{\mu B(x_{i_j}, s_{i_j})} \mathbf{1}_{Bl(x_{i_j}, s_{i_j}, ts_{i_j})}.$$

Integrating both sides and using microblossoming we conclude that  $\mu \cup \mathcal{C} \leq K \sum_i^N \mu D_{i_j} = K \mu \cup_{j=1}^N B(x_{i_j}, ts_{i_j})$ , whence  $\mu U \leq (K+1) \mu \cup_{j=1}^N B(x_{i_j}, ts_{i_j})$ .

Next we show that

$$\sum_{j=1}^N \frac{\mu D_{i_j}}{\mu B(x_{i_j}, s_{i_j})} \mathbf{1}_{B(x_{i_j}, s_{i_j})} \leq C(\mu) K + 1.$$

Suppose  $\sum_{j=1}^N \frac{\mu D_{i_j}}{\mu B(x_{i_j}, s_{i_j})} \mathbf{1}_{B(x_{i_j}, s_{i_j})}(z) > 0$ . Let  $\{B(x_{i_{k_1}}, s_{i_{k_1}}), \dots, B(x_{i_{k_n}}, s_{i_{k_n}})\}$  be the collection of all balls containing  $z$  (keeping the original ordering by decreasing radii). Then each  $B(x_{i_{k_j}}, s_{i_{k_j}})$  has radius either equal to or (substantially) larger than  $s_{i_{k_n}}$ . We separate the contributions of these balls into two sums. Suppose  $B(x_{i_{k_1}}, s_{i_{k_1}}), \dots, B(x_{i_{k_m}}, s_{i_{k_m}})$  all have radii larger than  $s_{i_{k_n}}$ , while  $B(x_{i_{k_{m+1}}}, s_{i_{k_{m+1}}}), \dots, B(x_{i_{k_n}}, s_{i_{k_n}})$  have radii equal to  $s_{i_{k_n}}$ . Now for  $1 \leq j \leq m$ , by  $T$  lacunarity and the fact that  $Tt \geq 1$ , we have  $s_{i_{k_n}} \leq ts_{i_{k_j}}$ , so  $z \in B(x_{i_{k_j}}, s_{i_{k_j}})$  implies that  $x_{i_{k_n}} \in Bl(x_{i_{k_j}}, s_{i_{k_j}}, ts_{i_{k_j}})$ , whence

$$\sum_{j=1}^m \frac{\mu D_{i_{k_j}}}{\mu B(x_{i_{k_j}}, s_{i_{k_j}})} \mathbf{1}_{Bl(x_{i_{k_j}}, s_{i_{k_j}}, ts_{i_{k_j}})}(x_{i_{k_n}}) \leq 1,$$

and thus

$$\sum_{j=1}^m \frac{\mu D_{i_{k_j}}}{\mu B(x_{i_{k_j}}, s_{i_{k_j}})} \mathbf{1}_{B(x_{i_{k_j}}, s_{i_{k_j}})}(z) \leq 1.$$

Next, note that the sets  $D_{i_{k_{m+1}}}, \dots, D_{i_{k_n}}$  are all disjoint and contained in  $Bl(z, s_{i_{k_n}}, ts_{i_{k_n}})$ . By microblossoming and local comparability, for  $j = m+1, \dots, n$  we have

$$\mu \cup_{j=m+1}^n D_{i_{k_j}} \leq \mu Bl(z, s_{i_{k_n}}, ts_{i_{k_n}}) \leq K \mu B(z, s_{i_{k_n}}) \leq K C(\mu) \mu B(x_{i_{k_j}}, s_{i_{k_n}}).$$

It follows that

$$\sum_{j=m+1}^n \frac{\mu D_{i_{k_j}}}{\mu B(x_{i_{k_j}}, s_{i_{k_n}})} \mathbf{1}_{B(x_{i_{k_j}}, s_{i_{k_n}})}(z) \leq \frac{C(\mu) \mu Bl(z, s_{i_{k_n}}, ts_{i_{k_n}})}{\mu B(z, s_{i_{k_n}})} \leq C(\mu) K.$$

□

Denote by  $M_R$  the centered Hardy-Littlewood maximal operator, with the additional restriction that the supremum is taken over radii belonging to the subset  $R \subset (0, \infty)$  (cf. [NaTa, p. 735]). We mention that under the hypotheses of the next corollary, it is not known whether the centered Hardy-Littlewood maximal operator  $M$  (with no restriction on the radii) is of weak type (1,1).



**Corollary 4.2.** *Let  $(X, d, \mu)$  be a metric measure space, where  $\mu$  satisfies a  $C(\mu)$  local comparability condition, and let  $R := \{r_n : n \in \mathbb{Z}\}$  be a  $T$ -lacunary sequence of radii. Suppose there exists a  $t > 0$  with  $Tt \geq 1$  such that  $\mu$   $(t, K)$ -microblossoms boundedly. Then  $\|M_R\|_{L^1-L^{1,\infty}} \leq (K+1) (C(\mu) K+1)$ .*

The proof is standard. We present it to keep track of the constants.

*Proof.* Fix  $\varepsilon > 0$ , let  $a > 0$ , and let  $f \in L^1(\mu)$ . For each  $x \in \{M_R f > a\}$  select  $B(x, r)$  with  $r \in R$ , such that  $a\mu B(x, r) < \int_{B(x, r)} |f|$ . Then the collection of “small” balls  $\{B(x, tr) : x \in \{M_R f > a\}\}$  is a cover of  $\{M_R f > a\}$ . By the  $\tau$ -smoothness of  $\mu$ , there is a finite subcollection  $\{B(x_i, ts_i) : s_i \in R, 1 \leq i \leq M\}$  of balls with positive measure, ordered by non-increasing radii, such that

$$(1 - \varepsilon)\mu\{M_R f > a\} \leq (1 - \varepsilon)\mu \cup \{B(x, tr) : x \in \{M_R f > a\}\} < \mu \cup_{i=1}^M B(x_i, ts_i).$$

Next, let  $\{B(x_{i_1}, s_{i_1}), \dots, B(x_{i_N}, s_{i_N})\}$  be the subcollection given by the Stein-Strömberg covering theorem for sparse radii. Then we have

$$\begin{aligned} \mu \cup_{i=1}^M B(x_i, ts_i) &\leq (K+1)\mu \cup_{j=1}^N B(x_{i_j}, ts_{i_j}) = (K+1) \sum_{j=1}^N \mu D_{i_j} \\ &= (K+1) \sum_{j=1}^N \frac{\mu D_{i_j}}{\mu B(x_{i_j}, s_{i_j})} \int \mathbf{1}_{B(x_{i_j}, s_{i_j})} \leq (K+1) \frac{1}{a} \int |f| \sum_{j=1}^N \frac{\mu D_{i_j}}{\mu B(x_{i_j}, s_{i_j})} \mathbf{1}_{B(x_{i_j}, s_{i_j})} \\ &\leq (K+1) (C(\mu) K+1) \frac{1}{a} \int |f|. \end{aligned}$$

□

In the specific case of  $d$ -dimensional Lebesgue measure  $\lambda^d$ ,  $C(\lambda^d) = 1$ . Choosing  $t = 1/d$  and  $T = d$ ,  $K$  above can be taken to be  $e^2$ , so the constant obtained is  $(e^2 + 1)^2$ , which is worse than the constant  $(e^2 + 1)(e + 1)$  yielded by the Stein-Strömberg argument. This discrepancy is due to the fact that our definition of microblossoming uses the uncentered blossom instead of the blossom, so from the assumption  $\mu(Blu(x, r, tr)) \leq K\mu B(x, r)$  we get the same bound  $\mu(Bl(x, r, tr)) \leq K\mu B(x, r)$  for the potentially smaller centered blossom. Of course, we could strengthen the definition, using blossoms, to obtain the same constant as in the Stein-Strömberg proof, but in the case of Lebesgue measure we prefer to consider it separately, using different values of  $(t, K)$  to lower the known bounds. We do this in the next section.

While Corollary 4.2 follows from the proof of the Stein-Strömberg covering theorem, it was not stated there but in [MeSo, Lemma 4] for Lebesgue measure, and in the microdoubling case, in [NaTa, Corollary 1.2]. A source of interest for this result comes from the fact that under  $(t, K)$ -microblossoming, the maximal operator defined by a  $(1+t)$ -lacunary set of radii  $R$  is controlled by the sum of  $N$  maximal operators with lacunarity  $1/t$ , where  $N$  is the least integer such that  $(1+t)^N \geq 1/t$ . Thus, the bound  $\|M_R\|_{L^1-L^{1,\infty}} \leq N(K+1) (C(\mu) K+1)$  follows. Under the additional assumption of  $(t, K^{1/2})$ -microdoubling, the maximal operator

defined by taking suprema of radii in  $[a, (1+t)a)$  is controlled by  $K^{1/2}$  times the averaging operator of radius  $(1+t)a$ . Putting these estimates together, and using the better bound for  $\mu B_l(x, r, tr) \leq K^{1/2} \mu B(x, R)$ , the following result due to Naor and Tao (cf. [NaTa, Corollary 1.2]) is obtained. Of course, in this case  $\mu$  is doubling and  $X$ , geometrically doubling.

**Corollary 4.3.** *Let  $(X, d, \mu)$  be a metric measure space, where  $\mu$  satisfies a  $C(\mu)$  local comparability condition and is  $(t, K^{1/2})$ -microdoubling. If  $N$  is the least integer such that  $(1+t)^N \geq 1/t$ , then*

$$\|M\|_{L^1-L^{1,\infty}} \leq N K^{1/2} (K+1) (C(\mu) K^{1/2} + 1).$$

This shows that the Stein-Strömberg covering theorem for sparse radii in metric spaces suffices to prove the Naor-Tao bounds, but no greater generality is achieved in either the spaces or the measures, since microdoubling is used in the last step. A second approach, which yields a slightly more general version of the result and gives better constants, consists in going back to the original Stein-Strömberg argument. Recall that when defining  $(t, K_1)$ -microblossoming, we set  $0 < t < 1$  and  $K_1 \geq 1$ . In the proof of the next result  $K := \max\{K_1, e\}$  is used to determine the size of the steps. For convenience we take  $K \geq e$ , but  $e$  is just one possible choice. Note that the condition on  $mri(r, T)$  below entails that  $\mu$  is doubling on its support, and hence  $\text{supp}(\mu)$  is geometrically doubling.

**Theorem 4.4. Stein-Strömberg covering theorem for bounded radii.** *Let  $(X, d, \mu)$  be a metric measure space such that  $\mu$  satisfies a  $C(\mu)$  local comparability condition, and is  $(t, K_1)$ -microblossoming. Set  $K = \max\{K_1, e\}$ . Let  $r > 0$ , and suppose there exists a  $T > 1$  such that  $K_2 := mri(r, T) < \infty$ . Let  $\{B(x_i, s_i) : r \leq s_i < Tr, 1 \leq i \leq M\}$  be a finite collection of balls with positive measure, given in any order, and let  $D_1 = B(x_1, ts_1), \dots, D_M = B(x_M, ts_M) \setminus \cup_1^{M-1} B(x_i, ts_i)$  be the disjointifications of the  $t$ -reduced balls. Then*

$$(10) \quad \sum_{i=1}^M \frac{\mu D_i}{\mu B(x_i, s_i)} \mathbf{1}_{B(x_i, s_i)} \leq C(\mu) K_1 K \left( 2 + \frac{\log K_2}{\log K} \right).$$

Since the big  $d \log d$  part in the estimates for the maximal operator (in  $\mathbb{R}^d$  with Lebesgue measure) comes from this case, which does not require any particular ordering nor any choice of balls, it is natural to enquire whether some additional selection process can lead to an improvement in the bounds. In general metric spaces this cannot be done, by [NaTa, Theorem 1.4], but it might be possible in  $\mathbb{R}^d$ . However, I have not been able to find such a new selection argument.

In the statement above,  $T$  is not assumed to be close to 1, and in fact it could be much larger than 2 (recall Example 3.5). From the viewpoint of the proof, the difference between  $T \gg 2$  and the assumption of  $t$ -microdoubling lies in the fact that the size of the steps will vary depending on the growth of balls, rather than having increments given by the constant factor  $1+t$  at every step. But the total number of steps will be determined by  $K$  and  $K_2$ , not by whether the factors are all equal to  $1+t$  or not.

*Proof.* Suppose

$$\sum_{i=1}^M \frac{\mu D_i}{\mu B(x_i, s_i)} \mathbf{1}_{B(x_i, s_i)}(y) > 0.$$

Let  $s = \min\{s_i : 1 \leq i \leq M \text{ and } y \in B(x_i, s_i)\}$ . Then  $r \leq s < Tr$ . Select

$$h_1 = \sup\{h > 0 : \mu B(y, (1+h)s) \leq K\mu B^{cl}(y, s) \text{ and } (1+h)s \leq Tr\}.$$

This is always possible since  $\lim_{h \downarrow 0} \mu B(y, (1+h)s) = \mu B^{cl}(y, s)$ . Now either  $(1+h_1)s = Tr$ , in which case the process finishes in one step, and then it could happen that  $\mu B^{cl}(y, (1+h_1)s) < K\mu B^{cl}(y, s)$ , or  $(1+h_1)s < Tr$ , in which case  $\mu B(y, (1+h_1)s) \leq K\mu B^{cl}(y, s) \leq \mu B^{cl}(y, (1+h_1)s)$  (the last inequality must hold, since otherwise we would be able to select a larger value for  $h_1$ ).

If  $h_2, \dots, h_m$  have been chosen, let

$$h_{m+1} := \sup\{h > 0 : \mu B(y, s(1+h)\prod_{i=1}^m(1+h_i)) \leq K\mu B^{cl}(y, s\prod_{i=1}^m(1+h_i)) \text{ and } s(1+h)\prod_{i=1}^m(1+h_i) \leq Tr\}.$$

Since  $\mu B(y, Tr) < \infty$ , the process stops after a finite number of steps, so there is an  $N \geq 0$  (assigning value 1 to the empty product) such that  $s\prod_{i=1}^{N+1}(1+h_i) = Tr$  and

$$\mu B^{cl}(y, s\prod_{i=1}^N(1+h_i)) \leq \mu B(y, Tr) \leq K\mu B^{cl}(y, s\prod_{i=1}^N(1+h_i)).$$

To estimate  $N$ , note that since  $r \leq s$ ,

$$\begin{aligned} \mu B(y, Tr) &\leq K_2 \mu B(y, s) \leq \frac{K_2}{K} \mu B^{cl}(y, (1+h_1)s) \\ &\leq \dots \leq \frac{K_2}{K^N} \mu B^{cl}(y, s\prod_{i=1}^N(1+h_i)) \leq \frac{K_2}{K^N} \mu B(y, Tr). \end{aligned}$$

Hence  $K^N \leq K_2$  and thus  $N \leq \log K_2 / \log K$ .

The remaining part of the argument is a variant of what was done in Stein-Strömberg for sparse radii, when considering the contribution of balls with the same radius as the smallest ball. Here we arrange the balls containing  $y$  into  $N+2$  “scales” (instead of just one) depending on whether their radii  $R$  are equal to  $s$ , or  $s\prod_{i=1}^m(1+h_i) < R \leq s\prod_{i=1}^{m+1}(1+h_i)$ , or  $s\prod_{i=1}^N(1+h_i) < R \leq Tr$ .

For the first scale, consider all balls  $B(x_{i_{1,1}}, s), \dots, B(x_{i_{1,k_1}}, s)$  containing  $y$ . Since for  $1 \leq j \leq k_1$ ,  $x_{i_{1,j}} \in B(y, s)$ , it follows that the disjoint sets  $D_{i_{1,j}}$  are all contained in  $Bl(y, s, ts)$ . By microblossoming and local comparability we have, for  $j = 1, \dots, k_1$ ,

$$\mu \cup_{j=1}^{k_1} D_{i_{1,j}} \leq \mu Bl(y, s, ts) \leq K_1 \mu B(y, s) \leq K_1 C(\mu) \mu B(x_{i_{1,j}}, s),$$

so

$$\sum_{j=1}^{k_1} \frac{\mu D_{i_{1,j}}}{\mu B(x_{i_{1,j}}, s_{i_{1,j}})} \mathbf{1}_{B(x_{i_{1,j}}, s_{i_{1,j}})}(y) \leq \frac{C(\mu) \mu Bl(y, s, ts)}{\mu B(y, s)} \leq C(\mu) K_1.$$

The contributions of all the other scales are estimated in the same way as the second one, which is presented next. Again, consider all balls  $B(x_{i_{2,1}}, s_{i_{2,1}}), \dots, B(x_{i_{2,k_2}}, s_{i_{2,k_2}})$  containing  $y$  and with radii  $s_{i_{2,j}}$  in the interval  $(s, (1+h_1)s]$ . Then all the sets  $D_{i_{2,j}}$  are contained in

$$Bl(y, (1+h_1)s, t(1+h_1)s).$$

Using microblossoming, the choice of  $h_1$ , and the local comparability of  $\mu$ , for  $j = 1, \dots, k_2$  we have

$$(11) \quad \begin{aligned} \mu \cup_{j=1}^{k_2} D_{i_{2,j}} &\leq \mu Bl(y, (1+h_1)s, t(1+h_1)s) \\ &\leq K_1 \mu B(y, (1+h_1)s) \leq K_1 K \mu B^{cl}(y, s) \leq K_1 K C(\mu) \mu B(x_{i_{2,j}}, s_{i_{2,j}}), \end{aligned}$$

so

$$\sum_{j=1}^{k_2} \frac{\mu D_{i_{2,j}}}{\mu B(x_{i_{2,j}}, s_{i_{2,j}})} \mathbf{1}_{B(x_{i_{2,j}}, s_{i_{2,j}})}(y) \leq \frac{C(\mu) \mu Bl(y, (1+h_1)s, t(1+h_1)s)}{\mu B^{cl}(y, s)} \leq C(\mu) K_1 K.$$

Adding up over the  $N+2$  scales we get (10).  $\square$

Next we put together the two parts of the Stein-Strömberg covering theorem. This helps to see why the original argument gives better bounds than domination by several sparse operators.

**Theorem 4.5. Stein-Strömberg covering theorem.** *Let  $(X, d, \mu)$  be a metric measure space, where  $\mu$  satisfies a  $C(\mu)$  local comparability condition, and is  $(t, K_1)$ -microblossoming. Set  $K = \max\{K_1, e\}$ , and suppose  $K_2 := \sup_{r>0} mri(r, 1/t) < \infty$ . Let  $\{B(x_i, s_i) : s_i \in R, 1 \leq i \leq M\}$  be a finite collection of balls with positive measure, ordered by non-increasing radii, and let  $U := \cup_{i=1}^M B(x_i, ts_i)$ . Then there exists a subcollection  $\{B(x_{i_1}, s_{i_1}), \dots, B(x_{i_N}, s_{i_N})\}$ , such that, denoting by  $D_{i_1} = B(x_{i_1}, ts_{i_1}), \dots, D_{i_N} = B(x_{i_N}, ts_{i_N}) \setminus \cup_{j=1}^{N-1} B(x_{i_j}, ts_{i_j})$ , we have*

$$(12) \quad \mu U \leq (K_1 + 1) \mu \cup_{j=1}^N B(x_{i_j}, ts_{i_j}),$$

and

$$(13) \quad \sum_{j=1}^N \frac{\mu D_{i_j}}{\mu B(x_{i_j}, s_{i_j})} \mathbf{1}_{B(x_{i_j}, s_{i_j})} \leq 1 + C(\mu) K_1 K \left( 2 + \frac{\log K_2}{\log K} \right).$$

*Proof.* The selection process is the same as in the proof of Theorem 4.1, yielding the desired subcollection, with (12) being the same as (8). As for the right hand side of (13) the 1 comes from the contribution of balls with very large radii, as in (9), while  $C(\mu) K_1 K \left( 2 + \frac{\log K_2}{\log K} \right)$  is the bound from (10).  $\square$

The same argument given for Corollary 4.2 now yields

**Corollary 4.6.** *Under the assumptions and with the notation of the preceding result, the centered maximal function satisfies the weak type (1,1) bound*

$$\|M\|_{L^1-L^{1,\infty}} \leq (K_1 + 1) \left( 1 + C(\mu) K_1 K \left( 2 + \frac{\log K_2}{\log K} \right) \right).$$

For Lebesgue measure on  $\mathbb{R}^d$ , with balls defined by an arbitrary norm and  $t = d^{-1}$ , this is worse (by a factor of  $e^2$ ) than the bound  $(1 + e^2)(1 + o(1))e^2 d \log d$  obtained by Stein and Strömberg.

Regarding lower bounds, currently it is known that for the centered maximal function defined using  $\ell^\infty$ -balls (cubes) the numbers  $\|M\|_{L^1-L^1, \infty}$  diverge to infinity (cf. [A]) at a rate at least  $O(d^{1/4})$  (cf. [IaSt]). No information is available for other balls. In particular, the question (asked by Stein and Strömberg) as to whether or not the constants  $\|M\|_{L^1-L^1, \infty}$  diverge to infinity with  $d$ , for euclidean balls, remains open.

## 5. SHARPENING THE BOUNDS FOR LEBESGUE MEASURE

Here we revisit the original case studied by Stein and Strömberg, Lebesgue measure  $\lambda^d$  on  $\mathbb{R}^d$ , with metric (and hence, with maximal function) defined by an arbitrary norm. Since  $\lambda^d$  is  $(t, (1+t)^d)$ -microdoubling for every  $t > 0$ , values of  $t \neq 1/d$  can be used to obtain improvements on the size of the constants.

**Theorem 5.1.** *Consider  $\mathbb{R}^d$  with Lebesgue measure  $\lambda^d$  and balls defined by an arbitrary norm. Let  $R := \{r_n : n \in \mathbb{Z}\}$  be a  $d$ -lacunary sequence of radii, and let  $M_R$  be the corresponding (sparsified) Hardy-Littlewood maximal operator. Then  $\|M_R\|_{L^1-L^1, \infty} \leq (e^{1/d} + 1)(1 + 2e^{1/d})$ . Furthermore, if the maximal function is defined using the  $\ell_\infty$ -norm, so balls are cubes with sides perpendicular to the coordinate axes, then  $\|M_R\|_{L^1-L^1, \infty} \leq 6$ .*

As we noted above, using the original argument from [StSt] one obtains  $\|M_R\|_{L^1-L^1, \infty} \leq (e^2 + 1)(e + 1)$ .

*Proof.* Suppose, for simplicity in the writing, that  $r_{n+1} = dr_n$  (the case  $r_{n+1} \geq dr_n$  is proven in the same way). We apply the Stein Strömberg selection process with  $t = 1/d^2$  and microdoubling constant  $K = (1 + 1/d^2)^d < e^{1/d}$ . As before, given  $0 \leq f \in L^1$  and  $a > 0$ , we cover the level set  $\{M_R f > a\}$  almost completely, by a finite collection of “small” balls  $\{B(x_i, ts_i) : s_i \in R, 1 \leq i \leq M\}$  ordered by non-increasing radii, and such that  $a\mu B(x_i, s_i) < \int_{B(x_i, s_i)} f$ . From this collection we extract a subcollection  $\{B(x_{i_1}, ts_{i_1}), \dots, B(x_{i_N}, ts_{i_N})\}$  satisfying

$$\mu \cup_{i=1}^M B(x_i, ts_i) \leq (e^{1/d} + 1) \mu \cup_{j=1}^N B(x_{i_j}, ts_{i_j}) = (e^{1/d} + 1) \sum_{j=1}^N \mu D_{i_j}.$$

Next, we obtain the bound

$$\sum_{j=1}^N \frac{\mu D_{i_j}}{\mu B(x_{i_j}, s_{i_j})} \mathbf{1}_{B(x_{i_j}, s_{i_j})} \leq 2e^{1/d} + 1,$$

by considering  $z$  such that  $\sum_{j=1}^N \frac{\mu D_{i_j}}{\mu B(x_{i_j}, s_{i_j})} \mathbf{1}_{B(x_{i_j}, s_{i_j})}(z) > 0$ . Select the ball  $B$  with largest index that contains  $z$ . Since  $B$  belongs to the subcollection obtained by the Stein-Strömberg method, all balls containing  $z$  and with radii  $\geq d^2 r(B)$  (where  $r(B)$  denotes the radius of  $B$ ), contribute at most 1 to the sum. Next we have to consider two more scales, all the

balls with radius  $r(B)$ , and all the balls with radius  $dr(B)$ . By the usual argument (as in the proof of Theorem 4.1) each of these scales contributes at most  $e^{1/d}$  to the sum, so  $\|M_R\|_{L^1-L^{1,\infty}} \leq (e^{1/d} + 1)(1 + 2e^{1/d})$  follows. The result for cubes is obtained by letting  $d \rightarrow \infty$ , since in this case it is known that the weak type (1,1) norms increase with the dimension (cf. [AV, Theorem 2]).  $\square$

**Theorem 5.2.** *Consider  $\mathbb{R}^d$  with Lebesgue measure  $\lambda^d$  and balls defined by an arbitrary norm. If  $\varepsilon > 0$ , then  $\|M\|_{L^1-L^{1,\infty}} \leq (2 + 3\varepsilon)d \log d$  for all  $d = d(\varepsilon)$  sufficiently large.*

The bound from the proof of [StSt, Theorem 1] is  $\|M\|_{L^1-L^{1,\infty}} \leq e^2(e^2 + 1)(1 + o(1))d \log d$ .

*Proof.* Fix  $\varepsilon \in (0, 1)$ . Since  $(1 + d^{-1-\varepsilon})^d = 1 + d^{-\varepsilon} + O(d^{-2\varepsilon})$ , it follows that  $\lambda^d$  is  $(d^{-1-\varepsilon}, 1 + d^{-\varepsilon} + O(d^{-2\varepsilon}))$ -microdoubling. Note that if a ball  $B$  contains the center of a second ball of radius 1, and the latter ball is contained in  $(1 + d^{-1-\varepsilon})B$ , then the radius  $r_B$  of  $B$  must satisfy  $r_B \geq d^{1+\varepsilon}$ . Let  $L$  be any natural number such that  $(1 + d^{-1-\varepsilon})^L \geq d^{1+\varepsilon}$ . Taking logarithms to estimate  $L$ , and using  $\log(1 + x) > x - x^2$  for  $x$  sufficiently close to 0, we see that it is enough, for the preceding inequality to hold, to choose  $L$  satisfying  $L(d^{-1-\varepsilon} - d^{-2-2\varepsilon}) \geq (1 + \varepsilon) \log d$ , or,  $L \geq (1 + o(d^{-1}))(1 + \varepsilon)d^{1+\varepsilon} \log d$ . For the least such integer we will have

$$L \leq 1 + (1 + o(d^{-1}))(1 + \varepsilon)d^{1+\varepsilon} \log d.$$

Again we apply the Stein Strömberg selection process with  $t = d^{-1-\varepsilon}$ , covering a given level set  $\{Mf > a\}$  almost completely (up to a small  $\delta > 0$ ) by a finite collection of small balls  $\{B(x_i, ts_i) : s_i \in R, 1 \leq i \leq k\}$  ordered by non-increasing radii, and such that  $a\mu B(x_i, ts_i) < \int_{B(x_i, ts_i)} |f|$ . Using the Stein Strömberg algorithm, we extract a subcollection

$$\{B(x_{i_1}, ts_{i_1}), \dots, B(x_{i_N}, ts_{i_N})\}$$

satisfying

$$(14) \quad (1 - \delta)\mu\{Mf > a\} \leq (2 + d^{-\varepsilon} + O(d^{-2\varepsilon})) \sum_{j=1}^N \mu D_{i_j},$$

where the sets  $D_{i_j}$  denote the disjointifications determined by the above subcollection. To sharpen the usual uniform bound for

$$\sum_{j=1}^N \frac{\mu D_{i_j}}{\mu B(x_{i_j}, s_{i_j})} \mathbf{1}_{B(x_{i_j}, s_{i_j})},$$

we use the fact that the sets  $D_i$  are disjoint across different steps, and not just within the same step. More precisely, let  $z$  satisfy

$$(15) \quad \sum_{j=1}^N \frac{\mu D_{i_j}}{\mu B(x_{i_j}, s_{i_j})} \mathbf{1}_{B(x_{i_j}, s_{i_j})}(z) > 0.$$

Select the ball  $B$  with largest index that contains  $z$ . Since  $B$  belongs to the subcollection obtained by the Stein-Strömberg method, all balls containing  $z$  and with radii  $\geq d^{1+\varepsilon}r(B)$

contribute at most 1 to the sum. Next we consider the first two scales, since for all the others, the argument is the same as for the second.

Take all the balls with radii equal to  $r_B$ . In order to bound (15) from above, we suppose that  $(1 + d^{-1-\varepsilon})B$  is completely filled up with the sets  $D_i$  associated to balls with radii  $r_B$ , and hence, no  $D_j$  associated to a ball with larger radius intersects  $(1 + d^{-1-\varepsilon})B$ . When we consider the sum (15), but just for the balls with radius  $r_B$ , we obtain the upper bound  $(1 + d^{-1-\varepsilon})^d$ . For the second level, we consider all balls in the subcollection with radii in  $(r_B, (1 + d^{-1-\varepsilon})r_B]$ , and as before, we suppose that  $(1 + d^{-1-\varepsilon})^2B \setminus (1 + d^{-1-\varepsilon})B$  is completely filled up with the sets  $D_j$  associated to these balls. The estimate we obtain for this second level is  $(1 + d^{-1-\varepsilon})^d - 1 = d^{-\varepsilon} + O(d^{-2\varepsilon})$ . For balls with radii in  $((1 + d^{-1-\varepsilon})^k r_B, (1 + d^{-1-\varepsilon})^{k+1} r_B]$ ,  $0 \leq k < L$ , we use the same estimate. Adding up over all scales we obtain

$$\sum_{j=1}^N \frac{\mu D_{i_j}}{\mu B(x_{i_j}, s_{i_j})} \mathbf{1}_{B(x_{i_j}, s_{i_j})}(z) \leq 1 + 1 + d^{-\varepsilon} + O(d^{-2\varepsilon})$$

$$+ (1 + (d^{-\varepsilon} + O(d^{-2\varepsilon}))(1 + o(d^{-1}))(1 + \varepsilon)d^{1+\varepsilon} \log d) \leq (1 + O(d^{-\varepsilon}))(1 + \varepsilon)d \log d.$$

Multiplying this bound with the bound from (14) and adding an  $\varepsilon$  to absorb the big Oh terms, for  $d$  large enough we obtain  $\|M\|_{L^1-L^{1,\infty}} \leq (2 + 3\varepsilon)d \log d$ .  $\square$

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